GENERALIZATION OF WHITTAKER'S FORMULA FOR PERIODIC ORBITS IN THE CASE OF FIELDS WITH ARBITRARY LAW OF ATTRACTION

(OBOBSHCHENIE FORMULY UITTEKERA DLIA PERIODICHESKIKH ORBIT NA SLUCHAI POLEI S PROIZVOL'NYM ZAKONOM PRITIAZHENIIA)

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Whittaker [1] established a suggestion for plane periodic orbits, described by a material point under the action of Newtonian centers of attraction. According to this suggestion a certain singular integral, distributed on the surface bounded by the closed orbit, is equal to the number of the attraction centers minus two.

In the following, Whittaker's formula is generalized for more general force fields when the attraction force is a certain function of distance, as well as for the force fields when the attraction by any center is inversely proportional to the nth degree of the distance. Such fields, in particular, were considered by Routh [2] and Liapunov [3] in the stability studies of the three body problem.

1. Let a point $M(x, y)$ of unit mass ($m = 1$), moving in a plane force field created by s centers of attraction located at the points O_j ($j = 1, 2,\ldots, s$) with the potentials

$$V_j = -A_j / r_j^n \qquad (A_j > 0) \tag{1.1}$$

perform a periodic motion by moving on a certain closed orbit c. If the differential equation for the trajectory of plane motions [4] is used, and if an angle $\psi(x, y)$ formed by the velocity vector and the positive direction of the x-axis is introduced, then we obtain

$$d\psi = \frac{\partial \Phi}{\partial y} dx - \frac{\partial \Phi}{\partial x} dy \tag{1.2}$$

$$(\Phi = \ln \sqrt{2(h - V(x, y))})$$

where h is constant energy, and $V(x, y)$ is the potential of the force field formed by all attraction centers located inside as well as outside the orbit c.

It follows from Formula (1.2) that

$$\frac{1}{2\pi} \oint_{(c)} \frac{\partial \Phi}{\partial y} dx - \frac{\partial \Phi}{\partial x} dy = 1 \tag{1.3}$$

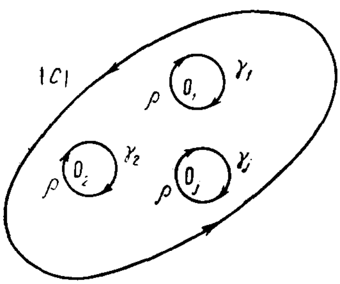

Fig. 1

Let us separate, inside the orbit c, small regions bounded by the circles γ_j ($j = 1, 2, \ldots, k$) with small radius ρ and with centers at O_j (Fig.1). In the thus obtained nonsimply connected region σ^* bounded by the contour Γ (contour Γ consistes of the contour c and the contours γ_j traversed so that the region σ^* reamains at left), the function

$$\Phi(x, y) = \ln \sqrt{2(h - V(x, y))}$$

will be continuous with a doubly continuous derivative and, consequently, on the strength of Green's theorem we get

$$\oint_{(c)} \frac{\partial\Phi}{\partial y}\,dx - \frac{\partial\Phi}{\partial x}\,dy - \oint_{(\gamma_1)} - \oint_{(\gamma_2)} - \ldots - \oint_{(\gamma_k)} = -\iint_{(\sigma^*)} \Delta\Phi(x, y)\,dx\,dy \qquad (1.4)$$

Decreasing the radii of the γ_j circles which enclose the attraction centers O_j, and passing to the limit for $\rho \to 0$ at the right-hand side in (1.4) we get an integral distributed over the entire surface σ bounded by the the contour c and, consequently, on the strength of (1.3) there results

$$2\pi - \sum_{j=1}^{k} J_j = -\iint_{(\sigma)} \Delta\Phi(x, y)\,dx\,dy \qquad (1.5)$$

where k is the number of attraction centers located inside the orbit, and

$$J_j = \lim_{\rho\to 0} \oint_{(\gamma_j)} \frac{\partial\Phi}{\partial y}\,dx - \frac{\partial\Phi}{\partial x}\,dy \qquad (1.6)$$

Let us express the expansion of the potential V in the neighborhood of the singular point O_j taken as the origin of the coordinates

$$V(r) = -A_j / r^n + V_1(r) \qquad (A_j > 0)$$

Here $V_1(r)$ is an integral function of r. Utilizing the expression for the function $\Phi(x, y)$ it is not difficult to obtain the expessions

$$\Phi_x = \frac{-nA_j x(1 + O(r^{n+1}))}{2r^{n+2}(h + A_j / r^n - V_1(r))} = -\frac{nx}{2r^2} + \ldots$$

$$\Phi_y = \frac{-nA_j y(1 + O(r^{n+1}))}{2r^{n+2}(h + A_j / r^n - V_1(r))} = -\frac{ny}{2r^2} + \ldots$$

where $O(r^{n+1})$ denotes terms of the r^{n+1} order of magnitude. Using these expansions we obtain

$$J_j = \lim_{r\to 0} \frac{n}{2r^2} \oint_{(\gamma_j)} (x\,dy - y\,dx) = \pi n \qquad (1.7)$$

and, consequently, on the strength of (1.5) we obtain the generalized Whittaker's formula

$$\frac{1}{2\pi} \iint_{(\sigma)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \ln\{h - V(x, y)\}\,dx\,dy = kn - 2 \qquad (1.8)$$

For the Newtonian attraction force field $n = 1$, and Formula (1.8) becomes Whittaker's formula. It is worth noting that Formula (1.8) is valid for the fractional values of n as well.

2. It is not difficult to see that Formula (1.8) also retains its form in the case when the potential V_j of the attraction center $O_j(x_j, y_j)$ is a more general expression

$$V_j = \frac{A_j}{r_j}\left(1 + \sum_{m=1}^{n-1} \frac{A'_{jm}}{r_j^{\,m}}\right) \qquad (2.1)$$

In particular, the potential of such a form, will be for the points lying in the equatorial plane of a spheroid if the known [5] expansion of the potential for the spheroid is an incomplete series.

Thus, for example, if the orbit of an equatorial satellite of the Earth is considered, then the potential V, accurate to terms of first order of magnitude with respect to Earth oblateness, is of the form $V = A/r + B/r^3$. The fields of the form (2.1) are also used in atomic physics where the electron m tion in the field of a nucleus is considered when studying the fine spectral structure [7].

For the potential $V(x, y)$ of a force field formed by all the centers of attraction O_j ($j = 1, 2, \ldots, s$) with the potentials V_j (2.1), the expansion in the neighborhood of some singular point O_j, taken as the origin of the coordinates, will be of the form

$$V(r) = \sum_{m=1}^{n} \frac{A_m}{r^m} + V_1(r) \tag{2.2}$$

where r is the distance from the point $M(x, y)$ to the center of attraction O_j, and $V_1(r)$ is an integral function of r. Proceeding as in the case for the attraction centers with potentials V_j (1.1), we again obtain the generalized Whittaker's formula (1.8).

3. Let us consider the fields formed by the logarithmic potentials

$$V_j = A_j \ln r_j \qquad (r_j = \sqrt{(x - x_j)^2 + (y - y_j)^2}) \tag{3.1}$$

Since for the potential $V = V_1 + \ldots + V_s$ the expansion in the neighborhood of some attraction center O_j which is taken as the origin of the coordinates, is of the form

$$V(r) = A_j \ln r + V_1(r) \tag{3.2}$$

where $V_1(r)$ is an integral function of r. Therefore

$$\Phi_x = \frac{-A_j x (1 + O(r))}{2r^2 (h - A_j \ln r - V_1(r))}, \qquad \Phi_y = \frac{-A_j y (1 + O(r))}{2r^2 (h - A_j \ln r - V_1(r))}$$

where $O(r)$ denotes terms of order of magnitude r, and the integral (1.6)

$$J_j = \lim_{r \to 0} \oint_{(\gamma_j)} \frac{A_j (x\,dy - y\,dx)}{2r^2 (h - A_j \ln r - V_1(r))} = \lim_{r \to 0} \frac{\pi A_j}{h - A_j \ln r - V_1(r)} = 0 \tag{3.3}$$

Consequently, on the strength of (1.5) and (3.3), the generalized Whittaker's formula becomes

$$\frac{1}{2\pi} \iint_{(\sigma)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \ln \{h - V(x, y)\}\, dx\, dy = -2 \tag{3.4}$$

i.e. for the case of a field formed by logarithmic potentials, the integral considered retains a constant value and is independent of the number k of the attraction centers O_j located inside the closed orbit c.

BIBLIOGRAPHY

1. Whittaker, E.T., On Periodic Orbits. Mon.Not.R.astr.Soc., Vol.62, pp.186-196, 1902.
2. Routh, E.J., On Laplace's Three Particles, with a Supplement on the Stability of Steady Motion. Proc.Lond.math.Soc., Vol.6, pp.86, 87, 1875.
3. Liapunov, A.M., Sobr.soch. (Collected Works). Vol.1, Izd.Akad.Nauk SSSR, p.378, 1954.
4. Belen'kii, I.M., Vvedenie v analiticheskuiu mekhaniku (Introduction to Analytical Mechanics). Izd. "Vysshaia shkola", pp.78-80, 1964.
5. Brauer, D. and Clemens, J., Metody nebesnoi mekhaniki (Methods of Celestial Mechanics). Izd. "Mir", p.115, 1964.
6. Beletskii, V.V., Orbita ekvatorial'nogo sputnika Zemli (Orbit of an Equatorial Satellite of the Earth). Izd.Akad.Nauk SSSR, Issue №13, 1962.
7. Zommerfeld, A., Stroenie atoma i spectry (Atomic Structure and Spectra). Vol.1, p.318, Gostekhizdat, 1956.

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